Network Algorithms and Dynamics

Homework 1 Due: 02/08/2017

1. Consider the Chernoff bound for the summation of n iid random variables $X_1, \cdots X_n$, i.e.,

$$P(\sum_{i=1}^{n} X_i \geq na) \leq e^{-nh(a)}.$$

Prove that $h(a) > 0$ if $a > E[X_1]$ and $E[e^{\theta X_1}] < \infty$ over $\theta \in [0, b]$ for some $b > 0$.

2. Find a Charnoff bound for the summation of n iid random variables $X_1, \cdots X_n$ in the form

$$P(\sum_{i=1}^{n} X_i \leq na) \leq e^{-n\tilde{h}(a)},$$

and identify $\tilde{h}(a)$.

3. Let X be the number of heads in a sequence of n independent fair coin flips, and $\mu = E[X] = \frac{n}{2}$. Apply the Chernoff bound to show that for any $\delta > 0$,

$$P(|X - \mu| > \delta\mu) \leq 2\exp(-\frac{\mu\delta^2}{3}).$$

Then use this bound to show that the deviation from the mean μ is of the order $O(\sqrt{n \log n})$.

4. Consider the exponentially tilted distribution $\{\hat{p}_k(s)\}$, $s \in \mathbb{R}$,

$$\hat{p}_k(s) = p_k \frac{s^k}{\phi(s)}, k = 0, 1, \cdots,$$

where $\phi(s)$ is the generating function of $\{p_k\}$. What is the mean of $\{\hat{p}_k(s)\}$?

5. Geometric branching process: Consider a Galton-Watson branching process with geometric offspring distribution with parameter p, i.e. $P(\xi = k) = p^k(1-p)$.

(i) Compute the probability of extinction occurring in generation n (using generating functions).

(ii) Give a general expression for the probability of extinction.

(iii) Derive an expression for the generating function of

$$W = \lim_{n\to\infty} \frac{X_n}{(E(\xi))^n},$$

where X_n is the number of individuals in generation n.