

Network Algorithms and Dynamics

Homework 4

Due: 04/03/2017

1. (Preferential attachment) Show that for large i , C_i (the fraction of nodes with degree i) can be approximated by $Ai^{-\frac{3-\alpha}{1-\alpha}}$ for a constant $A > 0$.
2. (Preferential attachment) Show that

$$\bar{X}_1(t+1) - \bar{X}_1(t) = 1 - \alpha \frac{\bar{X}_1(t)}{N(t)} - (1 - \alpha) \frac{\bar{X}_1(t)}{2E(t)}.$$

Recall that we used this result to calculate $\Delta_1(t)$ during the class.

3. (Doob's Martingale) Consider a sequence of random variables X_1, X_2, \dots, X_n , and let $M_i = \mathbb{E}[f(X_1, \dots, X_n) | X_1, \dots, X_i]$, $i = 0, 1, \dots, n$, for some real-valued function f . Using the properties of conditional expectation, show that M_i is indeed a martingale.
4. Consider a graph G with adjacency matrix A . Show that the number of paths of length t from node i to node j is the (i, j) -th entity of the matrix A^t .
5. For a symmetric matrix A show that $\|(I - \beta A)^{-1}\| = \rho((I - \beta A)^{-1}) = (1 - \beta \rho(A))^{-1}$, where $\|\cdot\|$ is the induced 2-norm and $\rho(\cdot)$ is the spectral radius.
6. Let X_i 's be iid bernoulli random variables ($p = 1/2$) and let $S_n = \sum_{i=1}^n X_i$.
 - a) Use the method of bounded difference and Azuma's inequality to get an upper bound on

$$P\left(\frac{S_n}{n} - \frac{1}{2} > \delta\right)$$

- b) Compare this with the upper bound obtained from the Chernoff bound.
7. Consider a ring graph with n nodes (i.e., each node i is connected to nodes $i - 1$ and $i + 1 \pmod{n}$), and a complete graph with n nodes. Suppose both networks start with 1 infected node under the SIR model and $\beta = \frac{\lambda}{n}$, $\lambda < 1$. Compare the size of nodes eventually removed. Which one is more vulnerable to the epidemic?