In the first lecture, we talked about Milgram’s experiment and “6 degrees of separation”. If we view the social world as a graph with edges indicating acquainted persons, then this graph must have a small diameter.

Which graph structures have this property?

ER graphs as we saw have small ($O(\log n)$) diameter. However, these graphs are not realistic models of social networks because social nets, typically consist of communities (i.e., friends in the same location, same school, same job, etc) so they are not totally random graphs.

Q: Can graphs with spatial structure have small diameter?

A: Yes! Strogatz - Watts ‘98

Consider a lattice of $n=m^2$ nodes, we have local edges as per the edges in the lattice. We also have “short-cut” edges generated as follows: each node $u$ creates a
short-cut edge to a uniformly random node with probability $p$. We denote the graph by SW($n$, $p$) without short-cuts: diameter = $2m = 2\sqrt{n}$ adding short-cuts significantly reduce the diameter.

![Graph showing short-cuts](image)

**Theorem:** There exists $A$ depending on $p$ s.t.

$$\lim_\limits{n \to \infty} \left( \text{D}(\text{SW}(n, p)) \leq A \log n \right) = 1$$

**Proof:** Refer to the book, ch. 6.

Milgram experiment not only revealed the small diameter property, but also contained an interesting algorithmic aspect: individuals were able to route the envelopes to their targets using only limited personal information at each step.

consider two nodes \( u, v \) on the lattice.

\[
|u-v| = |u_x-v_x| + |u_y-v_y|
\]

\( u = (u_x, u_y) \)

\( v = (v_x, v_y) \)

\( q_f \): # of shortcuts that each node generates.

node \( u \) chooses another node \( v \) as the destination of shortcut

with probability

\[
\frac{|u-v|^{-\alpha}}{\sum_{w \neq u} |u-w|^{-\alpha}}
\]

\( \alpha > 0 \).

\( \alpha = 0, q_f = 1 \implies \) Smallworld - Watts model.

\( \alpha \) controls the range of shortcuts.

Greedy routing: a node \( u \) trying to reach a node \( v \), forwards the message to node \( u \) which has the closest distance to \( v \) among its grid and shortcut neighbors.

Then: suppose \( \alpha = 2 \), for any \( u, v \), let \( T_{\text{greedy}}(u,v) \) be the # of steps by the greedy routing to reach \( v \) from \( u \), then

\[
\mathbb{E}[T_{\text{greedy}}(u,v)] \leq O((\log n)^2).
\]

Proof:

[Diagram of a network with nodes and connections]
Let $U_j = \{ w : 2^j < |w-v| \leq 2^{j+1} \}$.

Suppose $u(t)$ belongs to $U_j$, then with probability larger than

$$\min_{u(t) \in U_j} \sum_{w : |w-v| \leq 2^j} \frac{1}{|u(t) - w|^{-2}} \sum_{u(t) \neq w} |u(t) - w|^{-2}$$

$u(t+1) \in U_k$ for some $k < j$.

$$\sum_{u(t) \neq w} |u(t) - w|^{-2} \leq \sum_{i=1}^{2m} (4i) i^{-2} \leq 4 \left( 1 + \int_1^{2m} \frac{1}{x} \, dx \right)$$

$$\leq 4 \left( 1 + \log 2m \right).$$

The numerator $\geq (2^{j+1} + 2^j)^{-2} \sum_{i=1}^{2^j} i \geq \frac{1}{36}$

so probability of moving from $U_j$ to $U_k$ for $k < j$ is at least

$$\frac{1}{4 \cdot 4 \left( 1 + \log 2m \right)}$$

$1 + j \leq \log_2 2m$, so

$$\mathbb{E}[T_{\text{grad}}(w,v)] \leq 144 \left( 1 + \log 2m \right) \frac{2^m}{\log 2}$$

$$= O\left( \log^2 n \right).$$

**Impaseability of efficient routing for $\alpha < 2$.**

Decentralized algorithm: routing decision made at step $t$ depends only on knowledge of $u(0), \ldots, u(t)$ and their shortcuts.
For $\alpha < 2$, no decentralized alg. (including greedy alg.) can perform efficiently, i.e., for most pairs $(U, V)$

$$\Theta\left[T_{\text{alg}}(U, V)\right] = \Omega\left(m^{\frac{1-\alpha}{3}}\right)$$

(recall definition of $\Omega$).

Why? Consider $V = \{w : |v - w| \leq C\}$ for some constant $C$.

Let $t = \epsilon C, \epsilon \in (0, 1)$.

If alg. is to reach $V$ in $t$ steps from $U$, then the last shortcut used by alg. must end in $V$ (otherwise it will take at least $C$ steps to navigate to $V$ through grid edges).

Failure $\equiv$ alg. does not discover a shortcut leading to $V$ in the first $t$ steps. From any node $w$, the probability that its shortcut reaches $V$ is

$$\frac{\sum_{v' : |v' - v| \leq C} |v' - v|^{-\alpha}}{\sum_{v' \neq w} |v' - w|^{-\alpha}}$$

numerator $\leq |V| \leq 3C^2$; denominator

$$\sum_{i=1}^{\frac{m}{2}} i^{-\alpha} \geq \int_{\frac{m}{2}}^{\frac{m}{2}} x^{-\alpha} dx$$

$$\geq m^{1-\alpha}/2^{3-\alpha}$$
\[ P(\text{short cut reaches } \mathcal{N}) \leq bC^2 \cdot m^{\alpha-2} \]

\[ P(\text{failure}) \geq 1 - q + \sup_{\mathcal{N}} \text{ (short cut generated from } \mathcal{W} \text{ reaches } \mathcal{N}) \geq 1 - q \leq C (bC^2 \cdot m^{\alpha-2}) \]

\[ \epsilon = \frac{1}{12C} \quad C = (m^{\alpha-2})^{1/3} \]

\[ \Rightarrow \quad P(\text{failure}) \geq \frac{1}{2} \]

\[ \Rightarrow \quad \mathbb{E}[T_{\text{alg}}(u,v)] \geq 8 \left( \frac{\alpha-2}{m} \right) \quad \text{for some } \delta > 0 \]

when \( |u-v| > m \frac{\alpha-2}{3} \) (which is satisfied with high probability for any random pair \( (u,v) \) as \( m \to \infty \).)

Impossibility of efficient routing for \( \alpha > 2 \).

In this case \( \mathbb{E}[T_{\text{alg}}(u,v)] \geq \delta \left( \frac{|u-v|}{m} \right) m^\gamma \)

when \( \gamma = \frac{\alpha-2}{\alpha-1} \), \( \delta : \mathbb{R}^+ \to \mathbb{R}^+ \) increasing function.

why? Probability that a short cut generated at \( \mathcal{W} \) reaches \( \mathcal{W}' \)

s.t. \( |w-w'| > d \) is at most

\[ \sum_{x=\delta d}^{\infty} \frac{4 \cdot 4 \cdot 4 \cdot 4}{x^{\alpha-2}} \leq 4 \int_{1}^{\infty} x^{-\alpha} \ dx = 4 \int_{1}^{\infty} x^{-\alpha} \ dx \]

Consider \( (u,v) \) with \( |u-v| > t d \), then failure happens if
Shortest tours that are all steps have length less than \( d \). So

\[
\mathbb{E} \left[ T_{\text{alg}}(u,v) \right] \geq \left( 1 - q^t \frac{4}{d-2} \right)^{d-2}
\]

\( \forall (u,v) \) choose \( t_d = \frac{|u-v|}{2} \) and

\[
q^t \frac{4}{d-2} d^{d-2} = \frac{1}{2}
\]

\[
\Rightarrow d = \left| u_v \right|^{d-1} \left( \frac{4q^t}{d-2} \right)^{d-1}, \quad t = \left| u_v \right|^{d-1} \frac{1}{2} \left( \frac{4q^t}{d-2} \right)^{d-1}
\]

so

\[
\ell(t) = 2^{\frac{1}{4}} \left( \frac{4q^t}{d-2} \right)^{\frac{1}{d-1}}
\]