Epidemics on General Graphs

So far we studied SIR & SIS models on complete graphs, where each node can interact with other nodes. As we explained, real social networks have spatial structures, so we are interested to investigate epidemic models on general graphs.

Model of social network:

- an undirected graph $G(V, E)$, $|V| = n$ - population size

- Adjacency matrix of graph $A = [a_{ij}]_{i,j}$

$$ a_{ij} = \begin{cases} 1, & (i, j) \in E \\ 0, & \text{ otherwise} \end{cases} $$

Fact: $A$ is a symmetric, nonnegative matrix so by Perron-Frobenius theorem, all eigenvalues of $A$ are real, and the eigenvalue with the largest absolute value is positive, and its associated eigenvector has nonnegative entries.

$$ \lambda = \max \{|\lambda_1|, \ldots, |\lambda_n|\} = \lambda_1 \Rightarrow \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n $$

Spectral radius $\lambda_1$ is the largest eigenvalue in our case.
SIR model: (time slots $t=0,1,\ldots$)

- each infected node attempts to infect each of its neighbors,

  infection is successful with probability $\beta$, independently of other infection attempts.

- each infected node is removed after one time slot.

  $X_v(t) := \mathbb{I}(v \text{ is infected at the beginning of time slot } t)$

  $Y_v(t) := \mathbb{I}(v \text{ is removed at time slot } t)$

$\mathbb{P}(\text{susceptible node } u \text{ is infected at time slot } t+1)$

  $= 1 - \prod_{v \in \mathcal{N}(u)} (1 - \beta X_v(t))$

$\mathcal{N}(u) := \text{neighbors of } u = \{v : (u,v) \in \mathcal{E}\}$.

Q: How the eventual number of removed nodes $\{Y_v(t)\}$ depends on the graph topology and $\beta$? Is there a threshold similar to complex graph case?

$|X(t)| = \sum_{i=1}^{n} X_i(t)$ ; $|Y(t)| = \sum_{i=1}^{n} Y_i(t)$.

A: starting from initial infected node $X(0)$, suppose $\beta < 1$,

then
$E[Y_r(\omega)] \leq \frac{1}{\beta^0} \sqrt{v} |X(0)|$

If $G$ is a regular graph (i.e., each node has the same number of neighbors),

$E[Y_r(\omega)] \leq \frac{1}{\beta^0} \left|\frac{1}{1-\beta^0} \right| |X(0)|$.

Why?

- Starting from initial infected node $u_0$, an arbitrary node $v$ gets infected at time $t^*$ with probability at most

$$P(X_v(t^*) = 1) \leq \sum_{u \sim_v u_0} \beta^t X_{u_0}(0) = \beta^t (A^t)_{u_0, u}$$

So

$$P(Y_v(\omega) = 1) \leq \sum_{t=0}^{\infty} \sum_{u \in V} (RA)^t_{u,v} X_u(0)$$

So

$$E[Y_r(\omega)] = \sum_v P(Y_v(\omega) = 1) \leq \sum_{t=0}^{\infty} t^T (RA)^t X(0)$$

vector all all all

$$E[Y_r(\omega)] \leq t^T (I-BA)^{-1} X(0)$$

\[ \leq ||t|| ||I-BA|| ||X(0)|| \]

Corresponding induced norm $\rightarrow$ Euclidean norm
$$\| (I - RA)^{-1} \| = P \left( (I - RA)^{-1} \right) = (I - \lambda P)^{-1}$$

$$\| X(\omega) \| = \sqrt{\sum X(\omega)^2} \quad \| Y \| = \sqrt{n}.$$

In the case of regular graph:

$$(I - RA)^{-1} = \sum_{i=1}^{n} \frac{1}{1 - \beta \lambda_i} x_i x_i^T$$

so $$E(Y(\omega)) = \sum_{i=1}^{n} \frac{1}{1 - \beta \lambda_i} \Pi^T x_i x_i^T X(\omega).$$

\[ x_i = \frac{p_i}{d} \text{ and } x_i = \frac{1}{\sqrt{n}} \Pi \text{, and } x_i, 2x_i, \ldots \text{ are orthogonal to } x_i. \]

so $$E(Y(\omega)) \leq \frac{1}{(1 - \beta \lambda_i)^n} \Pi^T \Pi \Pi^T X(\omega) = \frac{1}{1 - \beta d} |X(\omega)|.$$
- SIS model

- A graph $G$ with nodes $\{1, 2, \ldots, n\}$

- $X(i) = 1$ (node $i$ is infected at time $t$)

- Infected node return to susceptible state at rate $\delta$ ($\geq 1$).

- Each susceptible node becomes infected at rate $\beta$ times the 
  number of infected neighbors.

Examples: epidemic of influenza

- failure in information storage systems

Q: What is the time to recover from the epidemic? (might be infinite)

We have a Markov chain with state space $\{0, 1\}^n$

$q_t(x, y)$: transition rate

\[
q_t(x, y) = \begin{cases}
\beta (1 - x_i) \sum_{j \in \mathcal{A}} x_j & \text{if } y = x + e_i; \\
x_i & \text{if } y = x - e_i; \\
0 & \text{o.w.}
\end{cases}
\]

This: $A$: adjacency matrix of $G$

$\rho$: spectral radius of $A$

\[
X(0) = (X(i, 0))_{i=1, \ldots, n} \quad \text{initial condition}
\]
Then,
\[ \mathbb{P}(X(t) \neq 0) \leq \sqrt{n} \sum_{i=1}^{\infty} X_i(t) \exp((\beta \ell - 1) t) . \]

This shows that when \( \beta \ell < 1 \), the recovery is possible (i.e., Markov chain \( X(t) \) will eventually absorb to \( 0 \)). Let \( \tau \) be the absorption time, then
\[
\mathbb{E} \tau = \int_0^\infty \mathbb{P}(\tau \leq t) \, dt = \int_0^\infty \mathbb{P}(X(t) \neq 0) \, dt
\]
\[
\leq \int_0^\infty \min\left(1, n \exp\left(-\left(1 - \beta \ell \right) t\right)\right) \, dt
\]
\[
= t^* \cdot \int_{t^*}^\infty n \exp\left(-\left(1 - \beta \ell \right) t\right) \, dt ; \quad t^* = \frac{\log n}{1 - \beta \ell}
\]
\[
\leq t^* + \frac{n}{1 - \beta \ell} \exp\left(-\left(1 - \beta \ell \right) t^*\right)
\]
\[
= \frac{\log n + 1}{1 - \beta \ell} \quad \text{(fast extinction time)}
\]

Proof of theorem: Here, we only present a sketch of the proof.

The idea is to relate the SIS Markov chain to another birth–death process which is easy to analyze. Consider the following

\( \lambda \) birth process with transition rates
\[
\rho_\lambda(X, Y) = \begin{cases} \beta \sum_{j \in \Omega} X_j, & Y = X + e_i \\ X_i, & Y = X - e_i \\ 0, & \text{o.w.} \end{cases}
\]
Then it follows that \( P(X(t) \neq 0) \leq P(X_{\text{brw}}(t) \neq 0) \leq e^t \mathbb{E}[X_{\text{brw}}(t)] \)

i.e., in stochastic sense, \( X_{\text{brw}}(t) \) dominates \( X(t) \).

Then, \( \frac{d}{dt} \mathbb{E}[X_{\text{brw}}(t)] = \beta \sum_{j \neq i} \mathbb{E}[X_{\text{brw}}(t + \tau_{ij}) - \mathbb{E}[X_{\text{brw}}(t)] \]

in matrix form:

\[
\frac{d}{dt} \mathbb{E}[X_{\text{brw}}(t)] = \beta A \mathbb{E}[X_{\text{brw}}(t)] - \mathbb{E}[X_{\text{brw}}(t)]
\]

so

\[
\mathbb{E}[X_{\text{brw}}(t)] = e^{\beta t (A - I)} X(0)
\]

so

\[
P(X(t) \neq 0) \leq e^t \mathbb{E}[X_{\text{brw}}(t)] \leq \mathbb{E}[X_{\text{brw}}(t)]
\]

\[
= \sqrt{\frac{1}{n} \sum_{i=1}^{n} X_i(0) \exp(t(B - I) \tau_{ii})}
\]