

## Distributed Averaging

- a simple distributed algorithm
- A set of nodes (agents) :  $N$  nodes
- Communication constraints captured by the edges (directed) in a graph  $G(V, E)$ . We assume  $G$  is strongly connected.
- each node possesses a real value  $x_i(0)$ , at node  $i$ . Each node

does :

$$x_i(t+1) = \sum_{j \in \{i\} \cup \mathcal{N}_i} w_{ij} x_j(t) \quad t=1, 2, \dots$$

$w_{ij} > 0$  if  $(i, j) \in E$  and  $0$  otherwise.

$$\sum_{j \in \{i\} \cup \mathcal{N}_i} w_{ij} = 1$$

In matrix form:

$$X(t+1) = W X(t)$$

weight matrix  $W = [w_{ij}]$

-  $W$  is row-stochastic, nonnegative, and irreducible

$\downarrow$   
 $\sum_j w_{ij} = 1$

$\downarrow$   
 $G$  is strongly connected.

So from Perron-Frobenius Theorem, eigenvalues of  $W$  are s.t.

$$\lambda_1 = |\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_N|$$

It's easy to check that  $1$  is an eigenvector of  $W$  with

$\frac{1}{\sqrt{N}} [1 \dots 1]^T$  beign the corresponding eigenvector. It

follows from Gershgorin circle theorem that all the eigenvalues of a stochastic matrix have to be  $\leq 1$ .

Gershgorin's circle theorem: Consider a matrix  $A = [a_{ij}]_{1 \leq i, j \leq n}$ .

Let  $R_i = \sum_{j \neq i} |a_{ij}|$  and  $D_i = D(a_{ii}, R_i)$  be a closed

disk at center  $a_{ii}$  with radius  $R_i$ . Then all the eigenvalues

of  $A$  are confined within  $D_1 \cup \dots \cup D_n$ .

Hence,  $\lambda_1 = 1$  and  $v_1 = \frac{1}{\sqrt{N}} \mathbf{1}_N$ .

If we further ensure  $W$  is primitive, then

$$1 = \lambda_1 > |\lambda_2| \geq |\lambda_3| \dots \geq |\lambda_N|$$

↓  
strict  
inequality

Primitive matrix: means  $\exists t_0$  s.t.  $\forall t \geq t_0, W_{ij}^t > 0$  for all  $i, j$ .

- One way to ensure  $W$  is primitive is to add self-loops, i.e.,

$w_{ii} > 0$  for at least one  $i$ .

- Spectral Decomposition  $W = V \underbrace{\Delta V^{-1}}_U$  where

$$\Delta = \text{diag}(\lambda_1, \dots, \lambda_N)$$

$$V = [v_1 \dots v_N] \quad , \quad U = [u_1 \dots u_N]$$

eigenvectors      ↖      ↗

By iterating,

$$X(t) = W^t X(0)$$

$$= (V \Lambda V^T)^t X(0)$$

$$= V \Lambda^t V^T X(0)$$

$$= \sum_{i=1}^N \lambda_i^t v_i v_i^T X(0)$$

$$= v_1 v_1^T X(0) + \sum_{i=2}^N \lambda_i^t v_i v_i^T X(0)$$

as  $t \rightarrow \infty$  since  $|\lambda_i| < 1$

$$\rightarrow \frac{1}{\sqrt{N}} \mathbb{1}_N^T X(0)$$

as  $t \rightarrow \infty$

every node reaches consensus

$$\lambda_i(\infty) = \frac{1}{\sqrt{N}} \mathbb{1}_N^T X(0) \quad \forall i$$

If we further assume  $W$  is symmetric, then

$$U = V^{-1} = V^T \Rightarrow u_i = v_i = \frac{1}{\sqrt{N}} \mathbb{1}_N$$

$$\Rightarrow \lambda_i(\infty) = \frac{1}{\sqrt{N}} \mathbb{1}_N^T X(0) = \lambda_{\text{average}}$$

- Speed of convergence is determined by  $\lambda_2(W)$

Q: which one has a faster convergence? complete graph or ring graph?