Network Algorithm & Dynamics

Solution 1,

1. $X_1, \dots, X_n$: n iid random variables.

Chernoff bound : $\mathbb{P}\left(\sum_{i=1}^{n} X_i \geq an\right) \leq e^{-nh(a)}$; $h(a) = \sup_{\theta \geq 0} \left(\theta a - \log \mathbb{E}\left[e^{\theta X_1}\right]\right)$

We want to show that if $a > E[X_1]$ & $E[e^{\theta X_1}] < \infty$ over $\theta \in [0, b]$ for Some $b > 0$, then $h(a) > 0$.

Let's define $L(\theta, a) = \theta a - \log \mathbb{E}(e^{\theta X_1})$. First observe that $L(0, a) = 0$ $\Rightarrow h(a) \geq 0$. Also:

$$\left.\frac{\partial L(\theta, a)}{\partial \theta}\right|_{\theta=0^+} = a - \left.\frac{E[X_1 e^{\theta X_1}]}{E[e^{\theta X_1}]}\right|_{\theta = 0^+} = a - E(X_1) > 0 \quad (\text{By assumption})$$

So $\exists \theta > 0$ s.t. $L(\theta, a) > 0 \implies h(a) > 0$

Note that (right) derivative of $L(\theta, a)\big|_{\theta=0^+}$ exists because $E[e^{\theta X_1}] < \infty$ over some interval. $\blacksquare$

2. $X_1, \dots X_n$: n iid random variables. We want to find Chernoff bound for $\mathbb{P}\left(\sum_{i=1}^{n} X_i \leq na\right)$.

$$\mathbb{P}\left(\sum_{i=1}^{n} X_i \leq na\right) = \mathbb{P}\left(\theta \sum_{i=1}^{n} X_i \geq n\theta a\right) \; ; \; \theta \leq 0$$

$$= \mathbb{P}\left(e^{\theta \sum_{i=1}^{n} X_i} \geq e^{n\theta a}\right)$$

$$\overset{\text{Markov inequality}}{\leq} \frac{\mathbb{E}\left[e^{\theta \sum_{i=1}^{n} X_i}\right]}{e^{n\theta a}} \overset{iid}{=} \frac{\left[E[e^{\theta X_1}]\right]^n}{e^{n\theta a}}$$

$$= e^{-n\left(\theta a - \log \mathbb{E}\left[e^{\theta X_1}\right]\right)}$$

Define $\tilde{h}(a) = \sup\limits_{\theta \leqslant 0} \left(\theta a - \log \mathbb{E}\left[e^{\theta X_1}\right]\right)$. So we have:

$$\mathbb{P}\left(\sum_{i=1}^{n} X_i \leqslant na\right) \leqslant e^{-n\,\tilde{h}(a)} \quad \blacksquare$$

3. $\mu = E[X] = n/2$. And we can write: $X = \sum_{i=1}^{n} I_i$; $I_i = \begin{cases} 1 & i^{th} \text{ flip is head} \\ 0 & \text{o.w.} \end{cases}$

$$\mathbb{P}(|X-\mu| > \delta\mu) = \mathbb{P}\left((X > \mu(1+\delta)) \cup (X < \mu(1-\delta))\right)$$

disjoint event

$$= \mathbb{P}(X > \mu(1+\delta)) + \mathbb{P}(X < \mu(1-\delta))$$

$$= \mathbb{P}\left(\sum_{i=1}^{n} I_i > \left(\tfrac{1}{2} + \tfrac{\delta}{2}\right)n\right) + \mathbb{P}\left(\sum_{i=1}^{n} I_i < \left(\tfrac{1}{2} - \tfrac{\delta}{2}\right)n\right)$$

Chernoff bound

$$\leqslant e^{-n\,h\left(\frac{1}{2}+\frac{\delta}{2}\right)} + e^{-n\,\tilde{h}\left(\frac{1}{2}+\frac{\delta}{2}\right)}$$

Now we try to find $h\left(\frac{1+\delta}{2}\right)$ & $\tilde{h}\left(\frac{1-\delta}{2}\right)$.

$$h\left(\frac{1+\delta}{2}\right) = \sup_{\theta \geqslant 0} \left(\theta\left(\frac{1+\delta}{2}\right) - \log \mathbb{E}\left[\underline{e^{\theta I_1}}\right]\right)$$

$$= \sup_{\theta \geqslant 0} \left(\theta\left(\frac{1+\delta}{2}\right) - \log\left(\frac{e^{\theta \cdot 0} + e^{\theta \cdot 1}}{2}\right)\right) = \sup_{\theta \geqslant 0} \left(\theta\left(\frac{1+\delta}{2}\right) - \log\left(\frac{1+e^{\theta}}{2}\right)\right)$$

Take derivative & put it equal to zero.

$$\left.\frac{1+\delta}{2} - \frac{e^{\theta}/2}{\frac{1+e^{\theta}}{2}}\right|_{\theta=\theta^*} = 0 \Rightarrow e^{\theta^*} = \frac{1+\delta}{1-\delta} \quad (*)$$

Note that, since r.v. X varies from 0 to n, $\mathbb{P}(|X-\mu| > \delta\mu)$ for

$\delta \geq 1$ is zero (which is not interesting). So from now on we assume that $\delta < 1$ and is nonnegative! Then equ. ⊛ gives us a valid θ^* which is greater than zero. So:

$$h\left(\frac{1+\delta}{2}\right) = \theta^*\left(\frac{1+\delta}{2}\right) - \log\left(\frac{1+e^{\theta^*}}{2}\right) = \frac{1+\delta}{2}\log\left(\frac{1+\delta}{1-\delta}\right) - \log\frac{1}{1-\delta}$$

$$= \frac{1}{2}\left[(1+\delta)\log(1+\delta) + (1-\delta)\log(1-\delta)\right] \quad (**)$$

By doing similar calculation we can get:

$$h\left(\frac{1-\delta}{2}\right) = \frac{1}{2}\left[(1+\delta)\log(1+\delta) + (1-\delta)\log(1-\delta)\right] = h\left(\frac{1+\delta}{2}\right)$$

To continue, we can apply Taylor series for $\log(1+\delta)$ & $\log(1-\delta)$:

$$h\left(\frac{1+\delta}{2}\right) \overset{**}{=} \frac{1}{2}\left[(1+\delta)\left(\delta - \frac{\delta^2}{2} + \frac{\delta^3}{3} - \frac{\delta^4}{4} \cdots\right) + (1-\delta)\left(1 - \delta - \frac{\delta^2}{2} - \frac{\delta^3}{3} - \cdots\right)\right]$$

$$= -\sum_{n:\text{even}} \frac{1}{n}\delta^n + \sum_{n:\text{odd}} \frac{1}{n}\delta^{n+1}$$

$$= \sum_{n:\text{odd}} \left(\frac{1}{n} - \frac{1}{n+1}\right)\delta^{n+1} \geq \frac{1}{2}\delta^2$$

So we can write:

$$\mathbb{P}(|X-\mu| > \delta\mu) \leq 2e^{-n h\left(\frac{1+\delta}{2}\right)} \leq 2e^{-n\delta^2/2} = 2e^{-\mu\delta^2}$$

$$\leq 2e^{-\frac{\mu\delta^2}{3}}$$

To show that derivation from μ is of the order $O(\sqrt{n\log n})$, take $\delta = O\left(\sqrt{\frac{\log n}{n}}\right)$. So we have $\delta\mu = O(\sqrt{n\log n})$ and:

$$\exp\left(-\frac{\mu\delta^2}{2}\right) = \exp\left(-O(\log n)\right) = O\left(1/n\right)$$

So we have:

$$\mathbb{P}\left(|X-\mu| > O(\sqrt{n\log n})\right) \leqslant O\left(1/n\right) \xrightarrow{n\to\infty} 0.$$

4. $\hat{\xi} \sim \{\hat{P}_k(s)\} = \left\{P_k \dfrac{s^k}{\phi(s)}\right\}$

$$E[\hat{\xi}] = \sum_{k=0}^{\infty} k\hat{P}_k(s) = \sum_{k=0}^{\infty} kP_k\frac{s^k}{\phi(s)}$$

$$\Phi(s) = \sum_{k=0}^{\infty} s^k P_k \rightarrow \frac{d\Phi(s)}{ds} = \Phi'(s) = \sum_{k=0}^{\infty} k s^{k-1} P_k$$

$$E[\hat{\xi}] = \sum_{k=0}^{\infty} kP_k s\frac{s^{k-1}}{\Phi(s)} = \frac{s\Phi'(s)}{\Phi(s)}$$

5.i) $\mathbb{P}\{\text{extinction occurring in gene. } n\} = \mathbb{P}_{\text{ext. in } n}$

From lecture note: $\mathbb{P}^{(n)}_{\text{ext}} = \mathbb{P}(X_n = 0) = \mathbb{P}\{\text{extinction in } n \text{ or before } n\}$

$$\Rightarrow \mathbb{P}_{\text{ext in } n} = \mathbb{P}^{(n)}_{\text{ext}} - \mathbb{P}^{(n-1)}_{\text{ext}} = \Phi_n(0) - \Phi_{n-1}(0) \; ; \; \Phi_n(s) = \mathbb{E}(s^{X_n})$$

Recall that we have a recursive equ. for $\Phi_n(s)$. i.e:

$$\Phi_n(s) = \Phi_\xi(\Phi_{n-1}(s)).$$

$$\Phi_\xi(s) = \sum_{k=0}^{\infty} s^k P_k = \sum_{k=0}^{\infty} (sp)^k (1-p) = \frac{1-p}{1-sp}$$ (*) which needs $|sp| < 1$ for convergence.

Moreover; $\mathbb{E}(\xi) = \dfrac{p}{1-p}$ (**) for $0 \leqslant p < 1$. From now we assume $0 < p < 1$ to eliminate trivial case ($p=0$). And let's show $\mathbb{E}(\xi)$ with μ.

From ⊛: $1-\varphi_\xi(s) = \dfrac{p(1-s)}{1-sp} \Rightarrow \dfrac{1}{1-\varphi_\xi(s)} = \dfrac{1-sp}{p(1-s)}$ $\Longrightarrow$

From ⊛⊛: $\mu(1-s) = \dfrac{p(1-s)}{1-p} \Rightarrow \dfrac{1}{\mu(1-s)} = \dfrac{1-p}{p(1-s)}$

$$\frac{1}{1-\varphi_\xi(s)} - \frac{1}{\mu(1-s)} = \frac{p(1-s)}{p(1-s)} = 1 \quad \text{(I)}$$

We are using (I) to find a closed form formula for $\varphi_n(s)$.

let's apply (I) for $s = \varphi_{n-1}(s')$ then we have:

$$\frac{1}{1-\varphi_\xi(\varphi_{n-1}(s'))} - \frac{1}{\mu(1-\varphi_{n-1}(s'))} = \frac{1}{1-\varphi_n(s')} - \frac{1}{\mu(1-\varphi_{n-1}(s'))} = 1$$

$$\rightsquigarrow \frac{1}{1-\varphi_n(s)} = 1 + \frac{1}{\mu(1-\varphi_{n-1}(s))} = 1 + \frac{1}{\mu} + \frac{1}{\mu^2}\left(\frac{1}{1-\varphi_{n-2}(s)}\right)$$

$$= 1 + \frac{1}{\mu} + \frac{1}{\mu^2} + \cdots + \frac{1}{\mu^{n-1}} \cdot \frac{1}{1-\varphi_\xi(s)}$$

$$= 1 + \frac{1}{\mu} + \frac{1}{\mu^2} + \cdots + \frac{1}{\mu^{n-1}} + \frac{1}{\mu^n} \cdot \frac{1}{1-s}$$

$$= \begin{cases} n + \dfrac{1}{1-s} & \text{for } \mu = 1 \\[2ex] \dfrac{1-\left(\frac{1}{\mu}\right)^n}{1-\frac{1}{\mu}} + \dfrac{1}{\mu^n(1-s)} & \text{for } \mu \neq 1 \end{cases}$$

$$\Longrightarrow \varphi_n(s) = \begin{cases} 1 - \dfrac{1}{n + \frac{1}{1-s}} & p = 1/2 \\[2ex] 1 - \dfrac{1}{\frac{1-\mu^{-n}}{1-\mu^{-1}} + \frac{1}{\mu^n(1-s)}} & p \neq 1/2,\ 0 < p < 1 \end{cases}$$

Using the expression we found for $\varphi_n(s)$ @ $s=0$, we can find prob. of extinction in generation n.

$$\mathbb{P}_{\text{ext. in } n} = \begin{cases} \dfrac{1}{n} - \dfrac{1}{n+1} & \text{for } p=\tfrac{1}{2} \\[2ex] \dfrac{1}{\frac{1-\mu^{1-n}}{1-\mu^{-1}} - \mu^{1-n}} - \dfrac{1}{\frac{1-\mu^{-n}}{1-\mu^{-1}} - \mu^{-n}} & \text{for } p\neq \tfrac{1}{2} \ \& \ 0<p<1 \end{cases}$$

ii) P_{ext} is equal to smallest solution of $s=\varphi_\xi(s)$ for $s\in[0,1]$

$$s=\frac{1-p}{1-ps} \to ps^2-s+1-p=0 \Rightarrow s=\frac{1\pm|2p-1|}{2p} = \begin{cases} \dfrac{1-p}{p} \\ 1 \end{cases}$$

$$\leadsto P_{ext} = \begin{cases} \dfrac{1-p}{p} & p>\tfrac{1}{2} \quad (\mu>1) \\ 1 & p\leq \tfrac{1}{2} \quad (\mu\leq 1) \end{cases}$$

iii) $W=\lim_{n\to\infty} \dfrac{X_n}{(E(\xi))^n}$. Let's define $W_n = \dfrac{X_n}{(E(\xi))^n} = \dfrac{X_n}{\mu^n}$.

We want to find an expression for $\varphi_W(s)$. Start with finding $\varphi_{W_n}(s)$:

$$\varphi_{W_n}(s) = \mathbb{E}\left[s^{W_n}\right] = \mathbb{E}\left[s^{\frac{X_n}{\mu^n}}\right] = \mathbb{E}\left[\left(s^{1/\mu^n}\right)^{X_n}\right]$$

$$= \varphi_n\left(s^{1/\mu^n}\right)$$

$$= \varphi_\xi\left(\varphi_{n-1}\left(s^{1/\mu^n}\right)\right)$$

$$= \varphi_\xi\left(\varphi_{n-1}\left(\left(s^{1/\mu}\right)^{1/\mu^{n-1}}\right)\right)$$

$$= \varphi_\xi\left(\varphi_{W_{n-1}}\left(s^{1/\mu}\right)\right)$$

Now let n goes to infinity, So ω_n & $\omega_{n-1} \longrightarrow \omega$, then φ_{ω_n} & $\varphi_{\omega_{n-1}} \longrightarrow \varphi_\omega$. Therefore:

$$\varphi_\omega(s) = \varphi_\xi\left(\varphi_\omega\left(s^{1/\mu}\right)\right) \qquad \blacksquare$$