1. \[
P(X \leq Y) = \int_0^\infty P(X \leq Y | X = x) f_X(x) \, dx
\]
\[
= \int_0^\infty P(x \leq Y | X = x) f_X(x) \, dx
\]
\[
= \int_0^\infty P(x \leq Y) f_X(x) \, dx \quad \text{(because } X \text{ and } Y \text{ are independent)}
\]
\[
= \int_0^\infty e^{-\mu x} \lambda e^{-\lambda x} \, dx = \frac{\lambda}{\lambda + \mu}
\]

2. (a) One of the first two customers departs first which is then replaced by the third customer. From that point onward, the service times of the remaining customers are still independent exponential random variables with the same parameter (by the memoryless property of exponential random variable). Hence either of them will depart next with equal probability.

In mathematical language: let \( T_i \) be the service time of the \( i \)-th customer, \( i = 1, 2, 3 \).

\[
P(\text{the person is the last one to exit}) = P(T_3 > |T_1 - T_2|)
\]
\[
= \frac{\lambda}{\lambda + \lambda} = 0.5
\]

where the last equality follows from the fact that \(|T_1 - T_2|\) is still exponential with the same parameter and using the result of problem 1.

(b) \( \frac{1}{\lambda} = 2 \) minutes

\[
\mathbb{E}[T] = \mathbb{E}[\min(T_1, T_2)] + \mathbb{E}[T_3] = 1^{\min} + 2^{\min} = 3^{\min}
\]

*: According to question 2 of HW0, \( \min(T_2, T_1) \sim \text{Exp}(2\lambda) \)

3. We show that \( N(t) \) satisfies the first definition of Poisson process, with rate \( \lambda_1 + \lambda_2 \).

i) \( N(0) = N_1(0) + N_2(0) = 0 \)

ii) Let define \( S = \min(X_1, X_2) \) where \( X_1 \sim \text{Exp}(\lambda_1) \) and \( X_2 \sim \text{exp}(\lambda_2) \). Let \( N(t) = N_1(t) + N_2(t) = k \) and \( S = s \) then \( N(t+s) = N_1(t+s) + N_2(t+s) = k+1 \) because either \( N_1 \) or \( N_2 \) increases by 1 at time \( s \). From question 2 of HW0, \( S \sim \text{Exp}(\lambda_1 + \lambda_2) \). Since \( X_i's \ (i=1,2) \) are iid exponentially distributed, \( S \)'s are iid exponentially distributed.

4. \( N(t) \) is a Poisson process with rate \( \lambda \):

(a) \( P(N(15) = 2 | N(60) = 2) \):
\[
\mathbb{P}(N(15) = 2|N(60) = 2) = \frac{\mathbb{P}(N(15) = 2, N(60) = 2)}{\mathbb{P}(N = 60)} = \frac{\mathbb{P}(N(15) = 2, N(60) - N(15) = 0)}{\mathbb{P}(N = 60)}\\
= \frac{\mathbb{P}(N(15) = 2) \mathbb{P}(N(45) = 0)}{\mathbb{P}(N = 60)}\\
= \frac{1}{16}
\]

(b) \(\mathbb{P}(N(15) \geq 1|N(60) = 2):\)

\[
\mathbb{P}(N(15) \geq 1|N(60) = 2) = 1 - \mathbb{P}(N(15) = 0|N(60) = 2) = 1 - \frac{\mathbb{P}(N(15) = 0, N(60) - N(15) = 2)}{\mathbb{P}(N = 60)}\\
= 1 - \frac{\mathbb{P}(N(15) = 0) \mathbb{P}(N(45) = 2)}{\mathbb{P}(N = 60)}\\
= \frac{7}{16}
\]

5. Global Balance Equations (it is easier to draw a state transition diagram first)

\[
\frac{1}{2} \pi_s + \frac{1}{2} \pi_s = \frac{3}{8} \pi_r + \frac{3}{8} \pi_c\\
\frac{3}{8} \pi_c + \frac{3}{8} \pi_c = \frac{1}{2} \pi_s + \frac{3}{8} \pi_r\\
\frac{3}{8} \pi_r + \frac{3}{8} \pi_r = \frac{1}{2} \pi_s + \frac{3}{8} \pi_c\\
\pi_s + \pi_c + \pi_r = 1
\]

From 2nd and 3rd equations, \(\pi_c = \pi_r.\) Hence

\[
\pi_s = \frac{3}{11}\\
\pi_c = \pi_r = \frac{4}{11}
\]

6. (a)

![Figure 1: The Markov Chain of M/M/m Queue](image)

Detailed Balance Equations:
\[
\sum_{i=0}^{\infty} \pi_i = 1 \\
\lambda \pi_0 = \mu \pi_1 \\
\lambda \pi_1 = 2\mu \pi_2 \\
... \\
\lambda \pi_{m-1} = m\mu \pi_m \\
\lambda \pi_m = m\mu \pi_{m+1} \\
...
\]

Let \( \rho = \frac{\lambda}{\mu} \), then:

\[
\pi_i = \frac{\rho^i}{i!} \pi_0 \quad \text{for } i = 0 \text{ to } m \\
\pi_i = \left(\frac{\rho}{m}\right)^{i-m} \pi_m = \left(\frac{\rho}{m}\right)^{i-m} \frac{\rho^m}{m!} \pi_0 \quad \text{for } i = m+1, m+2, ...
\]

Then to find \( \pi_0 \):

\[
\sum_{i=0}^{\infty} \pi_i = 1 \Rightarrow \sum_{i=0}^{m} \frac{\rho^i}{i!} \pi_0 + \sum_{i=m+1}^{\infty} \left(\frac{\rho}{m}\right)^{i-m} \frac{\rho^m}{m!} \pi_0 = 1 \\
\Rightarrow \pi_0 = \frac{1}{\sum_{i=0}^{m} \frac{\rho^i}{i!} + \frac{\rho^m}{m! m} \frac{1}{1 - \frac{\rho}{m}}}
\]

* For \( \rho < m \), which is the stability condition.

(b)

\[ P(\text{a customer has to wait}) = P(\text{a customer sees m customer or more in the system}) = P(\text{m persons or more in the system}) = \sum_{i=m}^{\infty} \pi_i = 1 - \sum_{i=0}^{m-1} \pi_i = 1 - \pi_0 \sum_{i=0}^{m-1} \frac{\rho^i}{i!} \]

* By using PASTA.

(c)

\[ \pi_0 = \frac{1}{\sum_{i=0}^{\infty} \frac{\rho^i}{i!}} = e^{-\rho} \quad \pi_i = e^{-\rho} \frac{\rho^i}{i!} \]

Note that this is a Possion distribution with mean \( \rho \). Hence, the number of customers in the \( M/M/\infty \) system in steady state is simply a Poisson random variable with mean \( \rho = \frac{\lambda}{\mu} \).

Also, the system is stable for all \( \rho < \infty \). And, \( P(\text{a customer has to wait}) \rightarrow 0 \)