Solution 2.

1. \( \lambda P_{\text{ext}}(\lambda) \leq 1 \) for G-W process with \( \xi \sim \text{Poisson}(\lambda) \):

Assume \( \lambda > 1 \), (for \( \lambda = 1 \), \( P_{\text{ext}} = 1 \) \( \Rightarrow \lambda P_{\text{ext}}(\lambda) = 1 \). For \( \lambda < 1 \) we have strict inequality and reasoning is same.)

\[
P_{\xi}(s) = e^{-\lambda (1-s)} \quad \Rightarrow \quad P_{\xi}'(s) = -\lambda e^{-\lambda (1-s)} \quad \Rightarrow \quad P_{\xi}'(s) \bigg|_{s=P_{\text{ext}}(\lambda)} = -\lambda e^{-\lambda (1-P_{\text{ext}}(\lambda))}
\]

Recall that \( P_{\text{ext}}(\lambda) = P_{\xi}(P_{\text{ext}}(\lambda)) \)

\[
\Rightarrow \quad P_{\xi}'(P_{\text{ext}}(\lambda)) = \lambda P_{\text{ext}}(\lambda)
\]

So we should look at derivative of \( P_{\xi} \) \( \Rightarrow \) \( P_{\xi}' \) is strictly convex.

\[ P_{\xi}' < 1 \quad \Rightarrow \quad P_{\xi} \text{ is strictly convex.} \]

all points of curve of \( P_{\xi} \) are above tangent at each point. (\( P_{\xi}'(s) > 0 \))

Let's define \( F(s) = P_{\xi}(s) - s \) \( \Rightarrow \) \( F(1) = 0 \), \( F(P_{\text{ext}}) = 0 \), \( F(1) = 0 \) \( \Rightarrow \) \( F'(P_{\text{ext}}) < 0 \)

\& \( F'(1) > 0 \) \( \Rightarrow \) \( F'(P_{\text{ext}}) = P_{\xi}'(P_{\text{ext}}) = 1 - \lambda P_{\text{ext}}(\lambda) - 1 < 0 \quad \Rightarrow \quad \lambda P_{\text{ext}}(\lambda) > 1 \)
3. \( G(n, p) \), \( p = \frac{c \log n}{n} \).

Let's define \( x_{ij} \) as indicator variable to determine whether node \( i \) and \( j \) construct a component of size 2 or not. To avoid recounting, we will define \( x_{ij} \) for \( 1 \leq i < j \leq n \).

\[
X_{ij} = \begin{cases} 
1 & 1 \leq i < j \leq n, i, j \text{ construct a component of size 2} \\
0 & \text{o.w.}
\end{cases}
\]

\[
P(X_{ij} = 1) = p(1-p)^{n-2} (1-p)^2 = p(1-p)^{2n-4}
\]

\[
P(\text{at least one connected component of size 2}) = P\left(\bigcup_{1 \leq i < j \leq n} X_{ij} = 1\right) = \sum_{i=1}^{n} \sum_{j=i+1}^{n} P(X_{ij} = 1)
\]

\[
= \binom{n}{2} p(1-p)^{2n-4}
\]

\[
= \frac{n!}{2! (n-2)!} \left(\frac{c \log n}{n}\right)^2 (1 - \frac{c \log n}{n})^{2n-4}
\]

\[
\approx \frac{c(n-1)}{2} \log n \ e^{-2c \log n (1 - \frac{c \log n}{n})^{-4}}
\]

\[
\approx (1 - \frac{c \log n}{n})^{2n} \approx \left(1 - \frac{1}{n \log n}\right)^{n \log n} \approx e^{-2c \log n} \quad \text{as } n \to \infty
\]

\[
P(\text{...}) \leq \frac{c^n}{2} \log n \ e^{-2c \log n (1 - \frac{c \log n}{n})^{-4}} \to \frac{c^n}{n} \log n \to 0 \quad \text{if } -2c > 1.
\]

\[
P(\text{graph has no comp. of s. 2}) \to 0 \quad \text{as } n \to \infty \quad \text{if } c > \frac{1}{2}
\]
4. \( G(n, \lambda_n) \): mean # of triangles:

Let's define: \( N \): # of triangles. And \( X_{ijk} \) as indicator:

\[
X_{ijk} = \begin{cases} 
1 & \text{if } i \neq j \neq k \text{ construct a triangle.} \\
0 & \text{o.w.}
\end{cases}
\]

\[ P(X_{ijk} = 1) = \left( \frac{\lambda^3}{n^3} \right)^3 \]

\[ N = \sum_{i \neq j \neq k} X_{ijk} \implies \mathbb{E}(N) = \sum_{i \neq j \neq k} \mathbb{E}(X_{ijk}) = \binom{n}{3} \left( \frac{\lambda^3}{n^3} \right)^3 = \frac{n(n-1)(n-2)\lambda^3}{6n^3} \]

\[ \mathbb{E}(N) \to \frac{\lambda^3}{6} \text{ as } n \to \infty \]

\( X_{ijk} \)'s are dependent. However, we can use Stein-Chen method to rigorously show that \( N \) converges to Poisson distribution with mean \( \frac{\lambda^3}{6} \).

Roughly speaking, the influence of each \( X_{ijk} \) on \( N \) is small. Note that there are \( O(n^3) \) indicator variables which determine \( N \). However, for each \( X_{ijk} \) there are \( O(n) \) indicators that depend on \( X_{ijk} \). (For example, \( X_{123} \) has an effect on \( X_{ijk} \)'s whose two indices are 1, 2 or 3. In instance, there are \( (n-3) \) indicators \( X_{12k} \) which correspond to nodes that has one common edge with \( X_{123} \).

Hence, if \( X_{ijk} = 1 \), it changes \( N \) slightly \( \left( \frac{O(n)}{O(n^3)} \to 0 \text{ as } n \to \infty \right) \). Therefore, \( N \) converges to Poisson distribution with mean \( \frac{\lambda^3}{6} \).

(In other words, \( X_{ijk} \)'s are dependent but they are weakly dependent.)