"Network Algorithms & Dynamics"

Solution 4.

1) Show that $c_i \simeq A\, i^{-\frac{3-\alpha}{1-\alpha}}$ for a constant $A>0$.

we have $\frac{c_i}{c_{i-1}} \simeq 1 - \frac{1}{i}\,\frac{3-\alpha}{1-\alpha}$ and by recursion we can write:

$$c_i = c_1 \prod_{j=2}^{i} \frac{c_j}{c_{j-1}} \simeq \frac{2}{3+\alpha} \prod_{j=2}^{i} \left(1 - \frac{1}{j}\,\frac{3-\alpha}{1-\alpha}\right)$$

for j large enough

$$\simeq \frac{2}{3+\alpha} \prod_{j=2}^{i} \exp\left(-\frac{1}{j}\,\frac{3-\alpha}{1-\alpha}\right) = \frac{2}{3+\alpha} \exp\left(-\frac{3-\alpha}{1-\alpha} \sum_{j=2}^{i} 1/j\right)$$

$$\overset{*}{\simeq} \frac{2}{3+\alpha} \exp\left(-\frac{3-\alpha}{1-\alpha}\left[\log i - 1\right]\right) = \frac{2}{3+\alpha}\left(i^{-\frac{3-\alpha}{1-\alpha}} \exp\left(-\frac{3-\alpha}{1-\alpha}\right)\right)$$

$$= A\, i^{-\frac{3-\alpha}{1-\alpha}}, \text{ where } A \triangleq \frac{2}{3+\alpha} \exp\left(-\frac{3-\alpha}{1-\alpha}\right)$$

* $\sum_{j=1}^{i} \frac{1}{j} \simeq \log_e i$ for large i so $\sum_{j=2}^{i} 1/j \simeq \log_e i - 1$. (we can also omit "1" and approximate $\log_e i - 1 \simeq \log_e i$)

we can also prove it using induction on i and Taylor series.

$\blacksquare$

2) Show that

$$\bar{X}_1(t+1) - \bar{X}_1(t) = 1 - \alpha \frac{\bar{X}_1(t)}{N(t)} - (1-\alpha)\frac{\bar{X}_1(t)}{2E(t)}$$

$$E\left(X_1(t+1) - X_1(t) \mid F_t \right) = 0 \times \Pr\left(X_1(t+1) - X_1(t) = 0 \mid F_t \right) + 1 \times \Pr\left(X_1(t+1) - X_1(t) = 1 \mid F_t \right)$$

$$= 1 - \alpha \frac{X_1(t)}{N(t)} - (1-\alpha)\frac{X_1(t)}{2E(t)}$$

$$E\left(X_1(t+1) - X_1(t)\right) = E\left(E\left(X_1(t+1) - X_1(t) \mid F_t \right)\right) = 1 - \alpha \frac{\bar{X}_1(t)}{N(t)} - (1-\alpha)\frac{\bar{X}_1(t)}{2E(t)}$$

$\longrightarrow$ result $\blacksquare$

3) To prove that M_i is Martingale with respect to $\{X_i\}_{i \geq 1}$, we use "Tower Property" of conditional expectation, which says:

$$E\left(E(X \mid A) \mid A_0 \right) = E(X \mid A_0) \quad \text{if } A_0 \subseteq A$$

for $j \leq i$ we have:

$$E\left(M_i \mid X_1, \dots, X_j \right) = E\left(E\left(f(X_1, \dots, X_n) \mid X_1, \dots, X_i \right) \mid X_1, \dots, X_j \right)$$

$$\overset{*}{=} E\left(f(X_1, \dots, X_n) \mid X_1, \dots, X_j \right) = M_j$$

$*$ holds because $\{X_1, \dots, X_j\} \subseteq \{X_1, \dots, X_i\}$ for $j \leq i$.

$\blacksquare$

4) we will prove the claim by induction:

Basis: $t=1 \rightarrow A^t = A$. By definition of A, $A_{ij}=1$ if i & j are neighbors so there is a path length 1 from i to j. So the claim is true.

Induction: Assume that $(A^t)_{ij}$ is the number of path from i to j of length t. (Induction hypothesis).

we can write $A^{t+1} = A^t . A$ so $(A^{t+1})_{ij} = \sum_{k=1}^{n} A^t_{ik} \times A_{kj}$. Since A^t_{ik} is # of paths from i to k with length t, if k is neighbor of j then A_{kj} is 1 and $A^t_{ik} \cdot A_{kj}$ gives us # of paths from i to j with length t through node k! if A_{kj} is zero it means that we cannot acheive j from k. Summation over all k, gives us total # of paths from i to j.

(The intuition behind this is that if I want to go from i to j in $t+1$ steps I should first go to j's neighbors in t steps and then use one extra step to get to j.)

Claim is proved by using Basis & Induction step.

$\boxtimes$

5) For a symmetric Matrix A show that $\|(I-\beta A)^{-1}\| \overset{①}{=} \rho\left((I-\beta A)^{-1}\right) \overset{②}{=} (1-\beta\rho(A))^{-1}$

① we show that $(I-\beta A)^{-1}$ is symmetric.

A is symmetric $\Rightarrow$ $C = I - \beta A$ is obviously symmetric. Also we can exchange the order of "transpose" and "inverse" operations, namely for any reversible matrix X we have : $(X^T)^{-1} = (X^{-1})^T$ so:

$(C^{-1})^T = (C^T)^{-1} = C^{-1}$ (because C is symmetric) $\rightarrow$ C^{-1} is also symmetric.

$\Rightarrow (I-\beta A)^{-1}$ is symmetric.

② Since second norm of a matrix is positive, from now on we assume that $1-\beta\rho(A) > 0 \Rightarrow \beta\rho(A) < 1 \Rightarrow \beta\lambda(A) < 1^{*}$ for all eigenvalues of A. $(\lambda(A))$

Another fact is that if λ is eigen value of a matrix, X, then λ^{-1} is eigen value of X^{-1}. (It is easy to prove it.)

$\Rightarrow \lambda(C^{-1}) = (\lambda(C))^{-1}$ & we also have:

$|(I-\beta A) - \lambda I| = |(1-\lambda)I - \beta A| = 0 \Rightarrow \lambda(I-\beta A) = 1 - \beta\lambda(A) \overset{*}{>} 0$

$\rho\left((I-\beta A)^{-1}\right) = \max_i \left\{ \left| \lambda_i\left((I-\beta A)^{-1}\right) \right| \right\} = \max_i \left\{ \lambda_i\left((I-\beta A)^{-1}\right) \right\} =$

$\left[\min_i \left\{ \lambda_i(I-\beta(A)) \right\}\right]^{-1} = \left[\min_i \left\{ 1-\beta\lambda_i(A) \right\}\right]^{-1} = (1-\beta\rho(A))^{-1}$

$\blacksquare$

6) a) Take X_i's to be iid random variables (Bernouli) with $P_0 = P_1 = 1/2$.

and define $S_n = \sum_{i=1}^{n} X_i \Rightarrow \mathbb{E}\left(\frac{S_n}{n}\right) = 1/2$.

Also define $M_i = \mathbb{E}\left[\frac{S_n}{n} \mid X_1 \cdots X_i\right]$ (here $\frac{S_n}{n} = \frac{\sum^{n} X_i}{n} = f(X_1 \cdots X_n)$)

So we can write $M_0 = 1/2$, $M_n = \frac{S_n}{n}$, so

$P\left(\frac{S_n}{n} - 1/2 > \delta\right) = P(M_n - M_0 > \delta) \leq \exp\left(\frac{-\delta^2}{2\sum_{i=1}^{n} c_i^2}\right)$ By Azuma's Inequality

So the only remaining point will be finding c_i's.

$$\left| M_i - M_{i-1} \right| = \left| E\left(\frac{S_n}{n} \mid X_1 \cdots X_i\right) - E\left(\frac{S_n}{n} \mid X_1 \cdots X_{i-1}\right) \right| =$$

$$\left| \frac{1}{n}\sum_{j=1}^{i} X_j + \frac{1}{2n}(n-i) - \left(\frac{1}{n}\sum_{j=1}^{i-1} X_j + \frac{1}{2n}(n-i+1)\right) \right| = \left| \frac{1}{n}X_i - \frac{1}{2n} \right| \leq 1/2n \text{ for all } i.$$

$$\Rightarrow P\left(\frac{S_n}{n} - 1/2 > \delta\right) \leq \exp\left(\frac{-\delta^2}{2n\frac{1}{4n^2}}\right) = \exp(-2n\delta^2)$$

b) In HW1, Solution to second problem, I found the exact bound and showed that

$h\left(\frac{1+\delta'}{2}\right) \geq 1/2\, \delta'^2$. where $\delta' = 2\delta$ (in that problem we wanted to find a bound

for $P(X - \mu > \delta\mu) = P(X - \mu > \frac{\delta'}{2})$) $\Rightarrow h(1/2 + \delta) \geq 2\delta^2$

$$\Rightarrow e^{-n h(1/2+\delta)} \leq e^{-2n\delta^2}$$

$\Rightarrow$ Chernoff bound is tighter than Azuma's.

7) Both graphs are regular graphs. We are just able to compare the upper bound for size of nodes eventually removed.

$\mathbb{E}(Y(\infty)) \leqslant \frac{1}{1-\beta\rho} |X(0)|$ where $\rho = d$ (degree of regular graph)

Complete graph : $\rho = n-1$ $\to E(Y(\infty)) \leqslant \frac{1}{1-\left(\frac{n-1}{n}\right)\lambda} \xrightarrow{n\to\infty} \frac{1}{1-\lambda}$

Ring graph: $\rho = 2$ $\to E(Y(\infty)) \leqslant \frac{1}{1-\frac{2\lambda}{n}} \longrightarrow 1$

Complete graph is more vulnerable to epidemic. $\blacksquare$